

Topological superfluid in a Fermi-Bose mixture with a high critical temperature

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We show that a two-dimensional (2D) spin-polarised Fermi gas immersed in a 3D Bose-Einstein condensate (BEC) constitutes a very promising system to realise a $p_x + ip_y$ superfluid. The fermions attract each other via an induced interaction mediated by the bosons, and the resulting pairing is analysed with retardation effects fully taken into account. This is further combined with Berezinskii-Kosterlitz-Thouless (BKT) theory to obtain reliable results for the superfluid critical temperature. We show that both the strength and the range of the induced interaction can be tuned experimentally, which can be used to make the critical temperature approach the maximum value allowed by general BKT theory. Moreover, this is achieved while keeping the Fermi-Bose interaction weak so that three-body losses are small. Our results show that realising a topological superfluid with atomic Fermi-Bose mixtures is within experimental reach.

The quest for realising topological phases of matter is presently a very active research topic [1, 2]. Topological superconductors/superfluids are of particular interest, as they exhibit Majorana edge modes with possible applications for quantum computing [3]. In condensed matter systems, experimental evidence for Majorana modes have been reported in a number of one-dimensional materials [4–10]. Furthermore, Sr_2RuO_4 is believed to realise a 2D topological superconductor [11–14]. However, it is highly desirable to find other systems which allow for unambiguous realisation of a topological superfluid. Cold atomic gases are attractive candidates for this task, as they are devoid of impurities and are highly controllable. Fermi gases interacting via a p -wave resonance have been suggested to realise topological superfluids [15], but they are found to have very short lifetimes [16–19]. Other suggestions using quantum gases include proposals based on optical lattices [20–23], synthetic spin-orbit coupling [24–26], driven dissipation [27, 28], dipolar molecules [29] and mixed dimension Fermi-Fermi mixtures [30]. However, none of these proposals have been implemented experimentally so far, partly due to the prohibitively low superfluid critical temperatures in these systems.

In this paper we show that a 2D-3D Fermi-Bose mixture can realise a $p_x + ip_y$ topological superfluid with a high critical temperature. Spin-polarised fermions are confined to a 2D plane and interact via an attractive induced interaction mediated by density fluctuations in a weakly interacting 3D BEC. The superfluid transition of the 2D Fermi gas is investigated by first solving the mean-field gap equation, which has the Eliashberg form due to the frequency dependence of the induced interaction. Here the retardation effects are fully included and are found to be important. The superfluid density is then calculated and the critical temperature of the transition determined using BKT theory. To our knowledge, such a microscopic theory for the pairing of a Fermi gas based on a combination of Eliashberg and BKT theory has not been presented before in the literature. We further demonstrate that the strength as well as the range

of the induced interaction can be controlled experimentally. This tunability can be utilised to increase the critical temperature of the superfluid transition to the limiting value imposed by BKT theory. Importantly, this is achieved while keeping both the Fermi-Bose and the Bose-Bose interactions weak, which is necessary to minimise three-body losses. Our results suggest a roadmap for realising a long lived topological superfluid in atomic gases with a high critical temperature.

Model.— We consider a single layer of spin-polarised, non-interacting fermions with mass m_F and areal density n_F , immersed in a uniform, weakly interacting 3D Bose gas with particle mass m_B and density n_B . The interaction between the fermions and the bosons is modelled by $g\delta(\mathbf{r})$. Here the coupling strength is given by $g = 2\pi a_{\text{eff}}/\sqrt{m_r m_B}$, where $m_r = m_F m_B/(m_F + m_B)$ is the reduced mass and a_{eff} is the effective 2D-3D scattering length; the latter can be tuned to arbitrary values in atomic gases [31]. The 2D Fermi gas is assumed to have no effect on the Bose gas, as it is much smaller in size. In the temperature regime relevant to our study, the weakly interacting Bose gas forms a BEC and is well described by the Bogoliubov theory. The partition function of the 2D-3D mixture at temperature T is ($\hbar = k_B = 1$)

$$\mathcal{Z} = \int \mathcal{D}(\bar{a}, a) \int \mathcal{D}(b^*, b) e^{-S(\bar{a}, a; b^*, b)}, \quad (1)$$

where (a, \bar{a}) and (b, b^*) are Grassmann and complex fields describing the fermions and bosons respectively. The action consists of $S(\bar{a}, a; b^*, b) = S_F^0 + S_B + S_{\text{int}}$. Here $S_F^0 = \beta \sum_{\mathbf{p}_{\perp}, n} \bar{a}(p_{\perp}) (-i\omega_n + \xi_{\mathbf{p}_{\perp}}) a(p_{\perp})$ describes free fermions with in-plane momentum $\mathbf{p}_{\perp} = (p_x, p_y)$, where $\beta = 1/T$, $\omega_n = (2n + 1)\pi/\beta$ is a Fermi Matsubara frequency and $\xi_{\mathbf{p}_{\perp}} = \mathbf{p}_{\perp}^2/2m_F - \mu$ is the fermion spectrum relative to the chemical potential μ of the Fermi gas. We have defined $p_{\perp} \equiv (\mathbf{p}_{\perp}, i\omega_n)$. Within Bogoliubov theory, the action for the BEC is $S_B = \beta \sum_{\mathbf{p} \neq 0, \nu} \gamma^*(p) (-i\omega_{\nu} + E_{\mathbf{p}}) \gamma(p)$. Here $p \equiv (\mathbf{p}, i\omega_{\nu})$ with $\mathbf{p} = (p_x, p_y, p_z)$, $\omega_{\nu} = 2\nu\pi/\beta$ is a Bose Matsubara frequency, $E_{\mathbf{p}} = \sqrt{\epsilon_{\mathbf{p}}(\epsilon_{\mathbf{p}} + 2g_B n_B)}$ with $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m_B$ is

the Bogoliubov spectrum and $\gamma(p) = u_{\mathbf{p}}b(p) + v_{\mathbf{p}}b^*(-p)$ are the quasi-particle fields. The Bogoliubov amplitudes are $u_{\mathbf{p}}, v_{\mathbf{p}} = \sqrt{\frac{1}{2}[(\epsilon_{\mathbf{p}} + g_B n_B)/E_{\mathbf{p}} \pm 1]}$, where $g_B = 4\pi a_B/m_B$ with a_B being the boson scattering length. Finally, the Fermi-Bose interaction is given by

$$S_{\text{int}} = g\sqrt{\frac{n_B}{\mathcal{V}}}\beta \sum_{\mathbf{p} \neq 0, \nu} [b^*(p) + b(-p)] \rho(p_{\perp}) \\ = g\sqrt{\frac{n_B}{\mathcal{V}}}\beta \sum_{\mathbf{p} \neq 0, \nu} \sqrt{\frac{\epsilon_{\mathbf{p}}}{E_{\mathbf{p}}}} [\gamma^*(p) + \gamma(-p)] \rho(p_{\perp}), \quad (2)$$

where \mathcal{V} is the volume of the BEC and $\rho(q_{\perp}) = \sum_{\mathbf{p}'_{\perp}, m} \bar{a}(\mathbf{p}'_{\perp} - \mathbf{q}_{\perp}, i\omega_m - i\omega_{\nu}) a(\mathbf{p}'_{\perp}, i\omega_m)$. In Eq. (2) we have ignored a term describing scattering between the fermions and the bosons that are not in the condensate, since we focus on the weak Fermi-Bose interaction regime where effects of this term are negligible. In the case of a strong interaction, however, it is crucial to include such a term [32]. We note that the interaction in Eq. (2) does not conserve momentum of two scattering particles along z -direction due to confinement of the fermions.

Induced interaction.— Performing the integration over the boson fields in Eq. (1) we find the following effective action for the fermions

$$S_F(\bar{a}, a) = S_F^0(\bar{a}, a) + \frac{\beta}{2\mathcal{A}} \sum_{q_{\perp}} V_{\text{ind}}(q_{\perp}) \bar{\rho}(q_{\perp}) \rho(q_{\perp}), \quad (3)$$

where \mathcal{A} is the area of the Fermi gas and

$$V_{\text{ind}}(\mathbf{q}_{\perp}, i\omega_{\nu}) = \frac{g^2 n_B}{m_B} \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{q^2}{(i\omega_{\nu})^2 - E_{\mathbf{q}}^2} \quad (4)$$

is the induced interaction between the fermions mediated by the bosons. Apart from the additional integration over the z -component of the Bose momentum, the formula in Eq. (4) is similar in form to the induced interaction in a 3D-3D Fermi-Bose mixture [33, 34]. It follows from Eq. (4) that $V_{\text{ind}}(\mathbf{q}_{\perp}, i\omega_{\nu})$ is manifestly real and negative. Performing the integral in Eq. (4) we find

$$V_{\text{ind}}(\mathbf{q}_{\perp}, i\omega_{\nu}) = -n_B m_B g^2 \left[\left(\frac{1}{\kappa_+} + \frac{1}{\kappa_-} \right) + \frac{1}{\sqrt{1 - (\omega_{\nu}/g_B n_B)^2}} \left(\frac{1}{\kappa_+} - \frac{1}{\kappa_-} \right) \right], \quad (5)$$

where $\kappa_{\pm} = \sqrt{2m_B g_B n_B} \left[1 \pm \sqrt{1 - (\omega_{\nu}/g_B n_B)^2} \right] + \mathbf{q}_{\perp}^2$. Here \sqrt{z} denotes the root of the complex number z with a positive real part. The frequency dependence of the induced interaction reflects the fact that density oscillations in the BEC have a finite speed. The most important frequency for pairing is on the order of the Fermi energy $\varepsilon_F = k_F^2/2m_F$, and $\varepsilon_F/g_B n_B \sim (m_F/m_B)(v_F/c_0)^2$, where $v_F = k_F/m_F$ is the Fermi velocity and $c_0 =$

$\sqrt{g_B n_B/m_B}$ is the speed of sound in the BEC. This suggests that the frequency dependence of the induced interaction can be neglected only if $v_F/c_0 \ll 1$, i.e. when Bogoliubov phonons move at a much greater speed than the fermions. In this case we can set $\omega_{\nu} = 0$ in Eq. (5), and the induced interaction assumes the simple form $V_{\text{ind}}(\mathbf{q}_{\perp}) \simeq -2n_B m_B g^2 / \sqrt{\mathbf{q}_{\perp}^2 + 2/\xi_B^2}$, where $\xi_B = 1/\sqrt{8\pi n_B a_B}$ is the BEC coherence length. We point out that the induced interaction in Eq. (5) is much stronger than that in a 2D-3D Fermi-Fermi mixture with comparable physical parameters [30], due to the fact that a BEC is in general more compressible than a Fermi gas.

$p_x + ip_y$ pairing.— The attractive induced interaction gives rise to pairing between the fermions. To describe this, we introduce a pairing field $\Xi(p)$ via the standard Hubbard-Stratonovich transformation, which couples the Grassmann fields with frequency/momenta p and $-p$. Since all momenta are now 2D, the \perp sign previously used to distinguish 2D vectors will be dropped henceforth. Integrating out the Grassmann fields one finds

$$\mathcal{Z} = \int \mathcal{D}(\Xi^*, \Xi) e^{-S_{\text{eff}}(\Xi^*, \Xi)}. \quad (6)$$

Here the effective action is

$$S_{\text{eff}} = -\frac{1}{2} \text{Tr} \ln G^{-1} - \frac{\beta \mathcal{A}}{2} \sum_{p, p'} \Xi^*(p) V_{\text{ind}}^{-1}(p-p') \Xi(p') \quad (7)$$

and $G^{-1}(p)$ is the inverse Green's function

$$G^{-1}(p) = \beta \begin{bmatrix} -i\omega_n + \xi_{\mathbf{p}} e^{i\omega_n 0^+} & \frac{\Xi(p) - \Xi(-p)}{2} \\ \frac{\Xi^*(p) - \Xi^*(-p)}{2} & -i\omega_n - \xi_{\mathbf{p}} e^{-i\omega_n 0^+} \end{bmatrix} \quad (8)$$

where 0^+ is an infinitesimal positive number. The inverse matrix $V_{\text{ind}}^{-1}(p-p')$ is defined by $\sum_q V_{\text{ind}}^{-1}(p-q) V_{\text{ind}}(q-p') = \delta_{pp'}$. In Eq. (7), Tr denotes the trace over the 2×2 matrix $\ln G^{-1}(p)$ and the summation over p . The mean-field theory for the pairing is obtained from the stationary phase condition $\delta S_{\text{eff}}(\Xi^*, \Xi)/\delta \Xi^*(p)|_{\Xi_s} = 0$. Defining the gap as $\Delta(p) = [\Xi_s(p) - \Xi_s(-p)]/2$, we obtain

$$\Delta(\mathbf{p}, i\omega_n) = -T \sum_m \int \frac{d\mathbf{q}}{(2\pi)^2} V_{\text{ind}}(\mathbf{p} - \mathbf{q}, i\omega_n - i\omega_m) \\ \times \frac{\Delta(\mathbf{q}, i\omega_m)}{\omega_m^2 + \mathcal{E}^2(\mathbf{q}, i\omega_m)} \quad (9)$$

and

$$n_F = T \sum_n \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{-i\omega_n e^{i\omega_n 0^+} - \xi_{\mathbf{q}}}{\omega_n^2 + \mathcal{E}^2(\mathbf{q}, i\omega_n)}, \quad (10)$$

where $\mathcal{E}(\mathbf{q}, i\omega_m) = \sqrt{\xi_{\mathbf{q}}^2 + |\Delta(\mathbf{q}, i\omega_m)|^2}$. Equations (9) and (10) retain the full frequency dependence of the gap and thus constitute a type of Eliashberg theory [35]. When the frequency dependence of the induced interaction and the gap is ignored, Eq. (9) simplifies to the

standard BCS gap equation. However, we shall see that retardation effects are generally important.

The pairing between the spin-polarised fermions has $p_x \pm ip_y$ symmetry due to the rotational symmetry of the system [36]. We thus look for a solution to the gap equation of the form $\Delta(\mathbf{p}, i\omega_n) = \Delta_1(p, i\omega_n)e^{i\phi_{\mathbf{p}}}$, where $\phi_{\mathbf{p}}$ is the azimuthal angle of \mathbf{p} . It follows that only the p -wave component of the induced interaction is relevant, which is defined as

$$V_1(p, q; i\omega_\nu) = \int_0^{2\pi} \frac{d\varphi}{2\pi} V_{\text{ind}}(\mathbf{p} - \mathbf{q}; i\omega_n - i\omega_m) e^{-i\varphi}, \quad (11)$$

where $\varphi \equiv \phi_{\mathbf{p}} - \phi_{\mathbf{q}}$. Substituting the p -wave form of the gap parameter into the Eq. (9) one easily obtains an equation involving $V_1(p, q; i\omega_\nu)$ which determines the amplitude $\Delta_1(p, i\omega_n)$. Crucial to the feasibility of our numerical solution to this gap equation, we are able to obtain an *analytic* expression for $V_1(p, q; i\omega_\nu)$ in terms of the complete elliptic integrals of the first and second kind. However, this expression is complex and will be provided in the Supplementary Material.

A powerful attribute of the present system is that both the *strength* and the *range* of the induced interaction can be controlled experimentally. From Eq. (4), we see that the induced interaction is proportional to the second power of $(n_B a_B^3)^{1/3}$, which measures the strength of the Fermi-Bose interaction. This parameter will be kept small compared to unity corresponding to a weak Fermi-Bose interaction. In addition to $(n_B a_B^3)^{1/3}$, the induced interaction depends on the following three dimensionless, independent parameters: the Bose gas parameter $(n_B a_B^3)^{1/3}$, the ratio of the inter-particle distances $n_F^{1/2}/n_B^{1/3}$, and finally the mass ratio m_F/m_B . To gain some intuition, we plot in Fig. 1 the zero-frequency component of the p -wave interaction at the Fermi surface, $V_1(k_F, k_F)$, as a function of $n_F^{1/2}/n_B^{1/3}$ for a mixture of ^{40}K and ^7Li . We see that the magnitude of $V_1(k_F, k_F)$ generally exhibits a maximum at a certain value of $n_F^{1/2}/n_B^{1/3}$ for fixed values of $(n_B a_B^3)^{1/3}$ and $(n_B a_B^3)^{1/3}$. This non-monotonic behaviour can be understood as follows. In the limit $n_F^{1/2}/n_B^{1/3} \rightarrow \infty$, the density of the Bose gas vanishes compared to that of the Fermi gas, and the induced interaction vanishes as a result. On the other hand, as $n_F^{1/2}/n_B^{1/3} \rightarrow 0$, the density of the Bose gas increases and the BEC coherence length ξ_B decreases (albeit keeping the Bose gas parameter constant). Since ξ_B determines the range of the induced interaction, the latter eventually becomes short ranged when ξ_B becomes small compared to distances between the fermions. This in turn leads to a suppressed p -wave interaction as we see in Fig. 1. Now, the same argument also suggests that the magnitude of $V_1(k_F, k_F)$ monotonically decreases as $(n_B a_B^3)^{1/3}$ increases for fixed values of $n_F^{1/2}/n_B^{1/3}$, since such a variation leads to a steady decrease of the BEC coherence length. This in fact is consistent with what we

observe in Fig. 1.

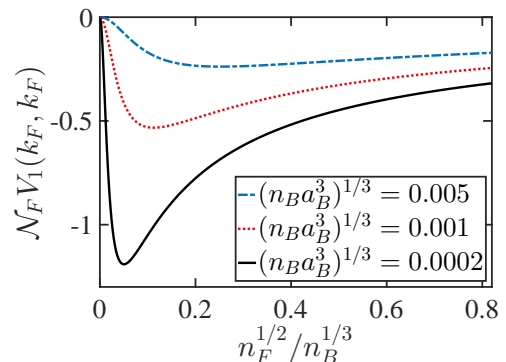


FIG. 1. The zero-frequency p -wave interaction at the Fermi surface as a function of $n_F^{1/2}/n_B^{1/3}$ for a weakly interacting ^{40}K - ^7Li mixture with $(n_B a_B^3)^{1/3} = 0.1$. Here $\mathcal{N}_F = m_F/2\pi$ is the density of states of the Fermi gas.

With all the frequency components of the p -wave induced interaction $V_1(p, q; i\omega_\nu)$ determined, the mean-field superfluid transition temperature T_{MF} can be obtained by solving the linearised gap equation. As an example, T_{MF} is shown in Fig. 2 for a mixture of ^{40}K and ^7Li as a function of the BEC gas parameter $(n_B a_B^3)^{1/3}$ for $n_F^{1/2}/n_B^{1/3} = 0.1$ and a weak Fermi-Bose interaction strength $(n_B a_B^3)^{1/3} = 0.1$. We see that the transition temperature increases with a decreasing $(n_B a_B^3)^{1/3}$ in agreement with the previous analysis concerning the strength of p -wave interaction. In fact, the mean-field transition temperature becomes significant compared to the Fermi energy for small $(n_B a_B^3)^{1/3}$. This is a promising result, even though phase fluctuations will reduce the critical temperature significantly as we shall see shortly. We also show the critical temperature T_{BCS} obtained from the BCS theory by neglecting the frequency dependence of the induced interaction, i.e., by using the $i\omega_\nu = 0$ component of the induced interaction in the gap equation. We see that these two temperatures indeed agree when $v_F/c_0 \ll 1$, as we have argued previously, whereas retardation effects significantly suppress the pairing for larger v_F/c_0 .

BKT transition temperature.— Since the Fermi system is 2D, phase fluctuations of the order parameter can significantly suppress the critical temperature of the superfluid transition. The transition is described by BKT theory, where the critical temperature is determined by the condition [37–40]

$$T_{\text{BKT}} = \frac{\pi}{8m_F^2} \rho_s (\{\Delta(i\omega_n)\}, T_{\text{BKT}}). \quad (12)$$

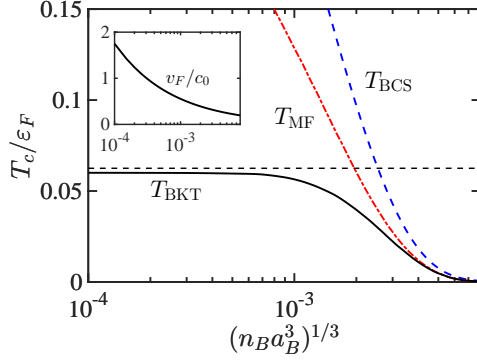


FIG. 2. The superfluid critical temperatures as a function of $(n_B a_B^3)^{1/3}$ (on a logarithmic scale) for the ^{40}K - ^7Li mixture with $n_F^{1/2}/n_B^{1/3} = 0.1$ and $(n_B a_{\text{eff}}^3)^{1/3} = 0.1$. The inset shows the behaviour of v_F/c_0 as a function of $(n_B a_B^3)^{1/3}$.

Here ρ_s is the superfluid mass density, which can be defined by considering the free energy density $F = \Omega + \mu n_F$ of the superfluid flowing at a velocity \mathbf{v} , where $\Omega = -T \ln \mathcal{Z}/\mathcal{A}$ is the grand potential density. For small velocities, we have $F(\mathbf{v}) - F(0) = \rho_s v^2/2$. Using $(\partial_v^2 F)_{n_F} = (\partial_v^2 \Omega)_\mu$ [41] and $\rho_s = \partial_v^2 F(\mathbf{v})|_{\mathbf{v}=0}$ [42, 43], the superfluid density can be obtained as $\rho_s = \partial_v^2 \Omega(\mathbf{v})|_{\mathbf{v}=0}$. Within mean field theory, we have $\Omega_{\text{MF}}(\mathbf{v}) = T S_{\text{eff}}(\mathbf{v})/\mathcal{A}$, where $S_{\text{eff}}(\mathbf{v})$ can be obtained from Eq. (7) by means of a momentum boost of $\mathbf{q} = m_F \mathbf{v}$. The momentum boost enters only in the diagonal components of $G^{-1}(p)$ in Eq. (8), which now read $-i\omega_n + \xi_{\mathbf{p}+\mathbf{q}} e^{i\omega_n 0^+}$ and $-i\omega_n - \xi_{\mathbf{p}-\mathbf{q}} e^{-i\omega_n 0^+}$. A straightforward evaluation of $\partial_v^2 \Omega_{\text{MF}}(\mathbf{v})|_{\mathbf{v}=0}$ yields (see Supplementary Material)

$$\rho_s = \rho_0 + \frac{T}{2} \sum_n \int \frac{d\mathbf{p}}{(2\pi)^2} p^2 \frac{\mathcal{E}^2(\mathbf{p}, i\omega_n) - \omega_n^2}{[\omega_n^2 + \mathcal{E}^2(\mathbf{p}, i\omega_n)]^2}, \quad (13)$$

where $\rho_0 = m_F n_F$. In the case of frequency-independent gap parameters $\Delta_{\mathbf{p}}$, Eq. (13) reduces to the well-known result $\rho_s = \rho_0 - (8T)^{-1} \int d\mathbf{p} (2\pi)^{-2} p^2 \text{sech}^2(\mathcal{E}_{\mathbf{p}}/2T)$.

The BKT transition temperature can now be obtained by solving Eqs. (9), (12) and (13) self-consistently [44]. The superfluid mass density ρ_s equals the total Fermi mass density ρ_0 at $T = 0$ and gradually decreases when T increases. We find that for a small Bose gas parameter, which corresponds to a long range induced interaction, the reduction in $\rho_s(T)$ by increasing temperature is rather small and $\rho_s(T) \simeq \rho_0$ when the BKT melting condition Eq. (12) is fulfilled. This suggests that the T_{BKT} in such a scenario will be close to the *maximum value allowed by BKT theory*, i.e., $T_{\text{BKT}} = \epsilon_F/16$. We point out that this value is in fact within experimental reach [45]. In Fig. 2, the BKT transition temperature is shown as a function of $(n_B a_B^3)^{1/3}$ for the previously given physical parameters. The transition temperature indeed quickly reaches the limiting value $\epsilon_F/16$ as $(n_B a_B^3)^{1/3}$ decreases.

Importantly, this maximum value is reached for a weak Fermi-Bose coupling with $(n_B a_{\text{eff}}^3)^{1/3} = 0.1$, which means that possible three-body losses due to a dimer formation from a fermion and a boson are small. Finally we emphasise that our system is very flexible in the sense that high transition temperatures can be reached across a *broad range* of parameter space. This is shown in Fig. 3, where BKT temperatures for various settings of parameters, all within the weak Fermi-Bose interaction regime, are calculated. We see that transition temperatures in the vicinity of the BKT limiting value are achieved for various density ratios of the mixture. Such a high flexibility makes our system advantageous in comparison to other proposals to realise p -wave superfluid in cold atomic systems.

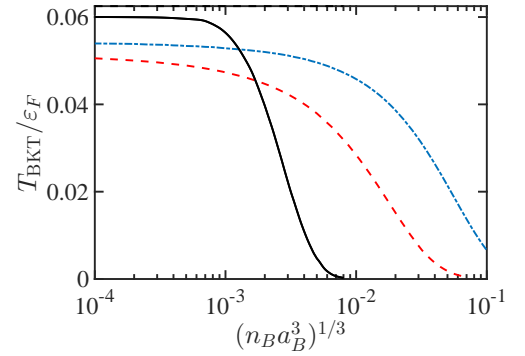


FIG. 3. The BKT temperatures as a function of $(n_B a_B^3)^{1/3}$ (on a logarithmic scale) for the ^{40}K - ^7Li mixture for various parameter settings, namely for (a) $n_F^{1/2}/n_B^{1/3} = 0.1$ and $(n_B a_{\text{eff}}^3)^{1/3} = 0.1$ (black solid), (b) $n_F^{1/2}/n_B^{1/3} = 0.5$ and $(n_B a_{\text{eff}}^3)^{1/3} = 0.15$ (red dashed), and (c) $n_F^{1/2}/n_B^{1/3} = 0.8$ and $(n_B a_{\text{eff}}^3)^{1/3} = 0.2$ (blue dash-dot).

Conclusions.— We have shown that a 2D-3D Bose-Fermi mixture is a promising system to realise a topological $p_x + ip_y$ superfluid. The fermions attract each other via density modulations in the BEC and form p -wave pairing. We analyse the pairing by solving the frequency-dependent gap equation which takes the retardation effects fully into account. The resulting Eliashberg theory was then combined with BKT theory to obtain a reliable microscopic theory for the superfluid critical temperature. Both the strength and the range of the induced interaction between the fermions can be controlled, and this can be used to tune the critical temperature to a limiting value imposed by BKT theory. Importantly, this maximum can be reached while keeping the Bose-Fermi interaction weak. Our results are directly relevant for experiments which use cold atomic gases to explore the topological superfluids.

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- [44] In principle the gap equation should be solved together with the particle number equation. However, the change of chemical potential due to paring is negligible for our calculations.
- [45] For example, the lowest temperature that can be achieved for a 3D Fermi gas at the ENS lab is at least $\varepsilon_F/20$ (private communication with F. Chevy).
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Supplemental Material

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THE p -WAVE COMPONENT OF THE INDUCED INTERACTION

To calculate $V_1(p, q; i\omega_\nu)$ defined in Eq. (11) we first introduce the dimensionless quantities $\bar{q} = q/k_F$ and $\bar{\omega}_\nu = \omega_\nu/\varepsilon_F$, where $\varepsilon_F = k_F^2/2m_F$ is the Fermi energy of the 2D gas. In terms of these quantities, $\mathcal{N}_F V_{\text{ind}}(\bar{\mathbf{p}} - \bar{\mathbf{q}}, i\bar{\omega}_\nu)$, where $\mathcal{N}_F = m_F/2\pi$ is the density of states of the Fermi gas, can be written as

$$\begin{aligned} \mathcal{N}_F V_{\text{ind}}(\bar{\mathbf{p}} - \bar{\mathbf{q}}, i\bar{\omega}_\nu) = & -\sqrt{\pi} \left(1 + \frac{m_F}{m_B}\right) \beta_{FB}^2 \left\{ \frac{1 - 1/\sqrt{1 - r_e^2 \bar{\omega}_\nu^2}}{\sqrt{2\beta_{BB} \left(1 - \sqrt{1 - r_e^2 \bar{\omega}_\nu^2}\right) + r_{FB}^2 [\bar{p}^2 + \bar{q}^2 - 2\bar{p}\bar{q} \cos(\phi_{\bar{\mathbf{p}}} - \phi_{\bar{\mathbf{q}}})]}} \right. \\ & \left. + \frac{1 + 1/\sqrt{1 - r_e^2 \bar{\omega}_\nu^2}}{\sqrt{2\beta_{BB} \left(1 + \sqrt{1 - r_e^2 \bar{\omega}_\nu^2}\right) + r_{FB}^2 [\bar{p}^2 + \bar{q}^2 - 2\bar{p}\bar{q} \cos(\phi_{\bar{\mathbf{p}}} - \phi_{\bar{\mathbf{q}}})]}} \right\} \end{aligned} \quad (14)$$

where $\beta_{FB} \equiv (n_B a_{\text{eff}}^3)^{1/3}$, $\beta_{BB} \equiv (n_B a_B^3)^{1/3}$, $r_{FB} \equiv n_F^{1/2}/n_B^{1/3}$ and $r_e \equiv \epsilon_F/g_B n_B = (m_B/m_F)r_{FB}/\beta_{BB}$. We see from the above expression that aside from the simple dependence on β_{FB} , the overall strength of the induced interaction is determined by three additional parameters, r_{FB} , β_{BB} and m_F/m_B .

Substituting Eq. (14) into Eq. (11) we find

$$\begin{aligned} \mathcal{N}_F V_1(\bar{p}, \bar{q}; i\bar{\omega}_\nu) = & -\sqrt{\frac{2}{\pi}} \left(1 + \frac{m_F}{m_B}\right) \frac{\beta_{FB}^2}{r_{FB} \sqrt{\bar{p}\bar{q}}} \left[\left(1 - \frac{1}{\sqrt{1 - r_e^2 \bar{\omega}_\nu^2}}\right) \frac{f_- K\left(\frac{2}{1+f_-}\right) - (1+f_-)E\left(\frac{2}{1+f_-}\right)}{\sqrt{1+f_-}} \right. \\ & \left. + \left(1 + \frac{1}{\sqrt{1 - r_e^2 \bar{\omega}_\nu^2}}\right) \frac{f_+ K\left(\frac{2}{1+f_+}\right) - (1+f_+)E\left(\frac{2}{1+f_+}\right)}{\sqrt{1+f_+}} \right], \end{aligned} \quad (15)$$

where

$$f_\pm = \frac{\beta_{BB}}{r_{FB}^2 \bar{p}\bar{q}} \left(1 \pm \sqrt{1 - r_e^2 \bar{\omega}_\nu^2}\right) + \frac{\bar{p}^2 + \bar{q}^2}{2\bar{p}\bar{q}}, \quad (16)$$

and $K(x)$ and $E(x)$ are the complete elliptic integral of the first and second kind. We note that for $|r_e \bar{\omega}_\nu| > 1$ the two terms inside the brackets of Eq. (15) are complex conjugates of each other so that $\mathcal{N}_F V_1(\bar{p}, \bar{q}; i\bar{\omega}_\nu)$ is always real. The zero frequency component of the expression in Eq. (15) at the Fermi surface is plotted in Fig. 1.

FREQUENCY DEPENDENT GAP EQUATION AND SUPERFLUID DENSITY

With the p -wave form of the gap parameter $\Delta(\mathbf{p}, i\omega_n) = \Delta_1(p, i\omega_n)e^{i\phi_{\mathbf{p}}}$, the gap equation (9) reduces to

$$\Delta_1(p, i\omega_n) = -\frac{1}{2\pi} T \sum_m \int dq q V_1(p, q; i\omega_n - i\omega_m) \frac{\Delta_1(q, i\omega_m)}{\omega_m^2 + \mathcal{E}(\mathbf{q}, i\omega_m)^2} \quad (17)$$

The mean-field critical temperature is determined by the linearised gap equation

$$\begin{aligned} \Delta_1(p, i\omega_n) = & -\frac{1}{2\pi} T \sum_m \int dq q V_1(p, q; i\omega_n - i\omega_m) \frac{\Delta_1(q, i\omega_m)}{\omega_m^2 + \xi_{\mathbf{q}}^2} \\ \equiv & \hat{L}(\Delta_1), \end{aligned} \quad (18)$$

where \hat{L} denotes the integral operator in the above equation. The mean-field critical temperature is obtained from the condition that the largest eigenvalue of \hat{L} becomes unity.

To determine the superfluid density we consider the grand potential of a flowing superfluid

$$\Omega_{\text{MF}}(\mathbf{v}, T, \mu) = \frac{T}{\mathcal{A}} S_{\text{eff}}(\mathbf{v}) = -\frac{T}{\mathcal{A}} \ln \mathcal{Z}_{\text{MF}}(\mathbf{v}), \quad (19)$$

where

$$\mathcal{Z}_{\text{MF}}(\mathbf{v}) = \det(-2\beta\mathcal{A}V_{\text{ind}}^{-1}) \exp \left\{ \frac{\beta\mathcal{A}}{2} \sum_{p,p'} \Delta^*(p) V_{\text{ind}}^{-1}(p-p') \Delta(p) \right\} \int \mathcal{D}(\bar{a}, a) \exp \left\{ - \sum_{\omega_n > 0, \mathbf{p}} \bar{\Lambda}(p) G^{-1}(p, \mathbf{q}) \Lambda(p) \right\}, \quad (20)$$

where $\mathbf{q} = m_F \mathbf{v}$, $\bar{\Lambda}(p) = [\bar{a}(p) \ a(-p)]$ and $\Lambda(p) = [a(p) \ \bar{a}(-p)]^T$. Here $G^{-1}(p, \mathbf{q})$ is given by

$$G^{-1}(p, \mathbf{q}) = \begin{bmatrix} -i\omega_n + \xi_{\mathbf{p}+\mathbf{q}} e^{i\omega_n 0^+} & \Delta(p) \\ \Delta^*(p) & -i\omega_n - \xi_{-\mathbf{p}+\mathbf{q}} e^{-i\omega_n 0^+} \end{bmatrix}. \quad (21)$$

One finds

$$\begin{aligned} \Omega_{\text{MF}}(\mathbf{v}, T, \mu) = & \text{const.} - \frac{1}{2} \sum_{p,p'} \Delta^*(p) V_{\text{ind}}^{-1}(p-p') \Delta(p) \\ & - \frac{T}{2\mathcal{A}} \sum_p \ln \left[- \left(-i\omega_n + \xi_{\mathbf{p}} e^{i\omega_n 0^+} \right) \left(-i\omega_n - \xi_{\mathbf{p}} e^{-i\omega_n 0^+} \right) + |\Delta(p)|^2 - R(v) \right] \end{aligned} \quad (22)$$

where

$$R(v) = -\frac{1}{4} m_F^2 v^4 + (\mathbf{p} \cdot \mathbf{v})^2 - m_F v^2 \xi_{\mathbf{p}} - \frac{1}{2} i\omega_n m_F v^2 \left(e^{i\omega_n 0^+} - e^{-i\omega_n 0^+} \right) - i\omega_n \mathbf{p} \cdot \mathbf{v} \left(e^{i\omega_n 0^+} + e^{-i\omega_n 0^+} \right) \quad (23)$$

Using $\rho_s = \partial_v^2 \Omega_{\text{MF}}(\mathbf{v})|_{\mathbf{v}=0}$ together with Eq. (22) we arrive at the expression in Eq. (13).